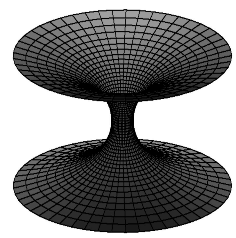
Monday October 5, 2020 -

Geometric & Functional Inequalities and Applications

Title: Heat kernel on manifolds with finitely many ends



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Joint work with A. Grigor'yan & S. Ishiwata.

$\Delta = -\frac{1}{\omega}\operatorname{div}(\omega \nabla)$

I Set up :

$(M, \omega dx)$ complete weighted Riemannian manifold (or a nice domain in $\mathbb{R}^n$, Neumann bdr cond)

$$\int_M g(\nabla f_1, \nabla f_2)\, \omega\, dx = \int_M f_1 \Delta f_2\, \omega\, dx$$

The heat equation: $(\partial_t + \Delta) u = 0$

The heat kernel $\quad p(t,x,y) \quad t>0,\ x,y \in M.$

II Harnack manifolds :

$d\mu = \omega dx$

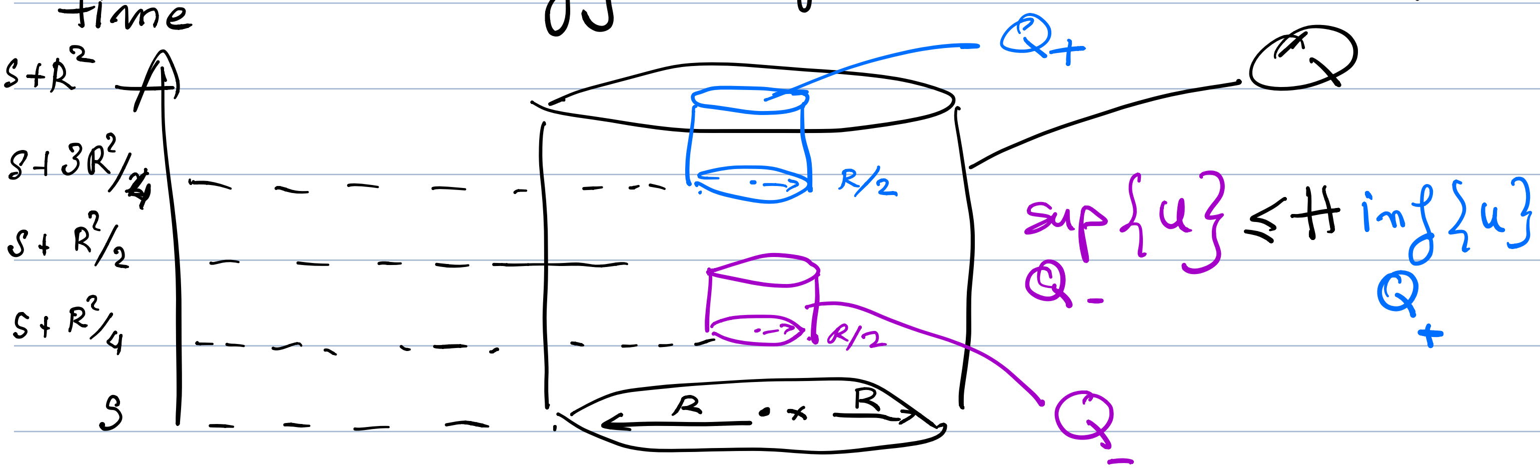
(1) $\begin{cases} \mu(B(x,2r)) \leq D\, \mu(B(x,r)) \\ \ldots \end{cases}$

$$\left(\int_B |f - f_B|\, d\mu \le P r \int_B |\nabla f|\, d\mu \right.$$

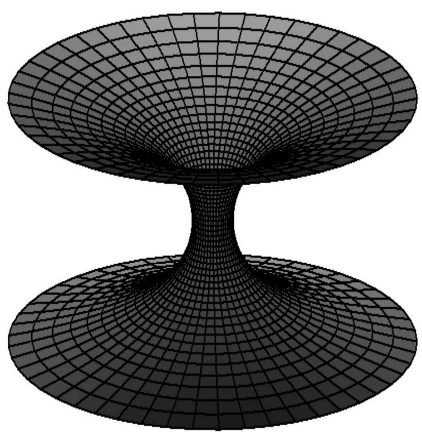
Set $V(x,r) = \mu(B(x,r))$.

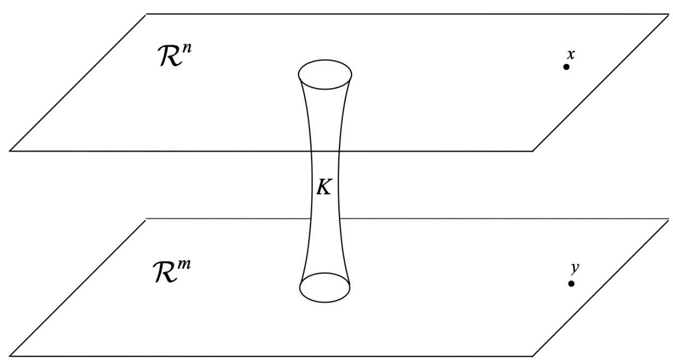
(2) $$p(t,x,y) \simeq \frac{1}{V(x,\sqrt{t})} \exp\left(-\frac{d(x,y)^2}{t}\right)$$

(3) positive solutions of the heat equation satisfy the parabolic Harnack inequality

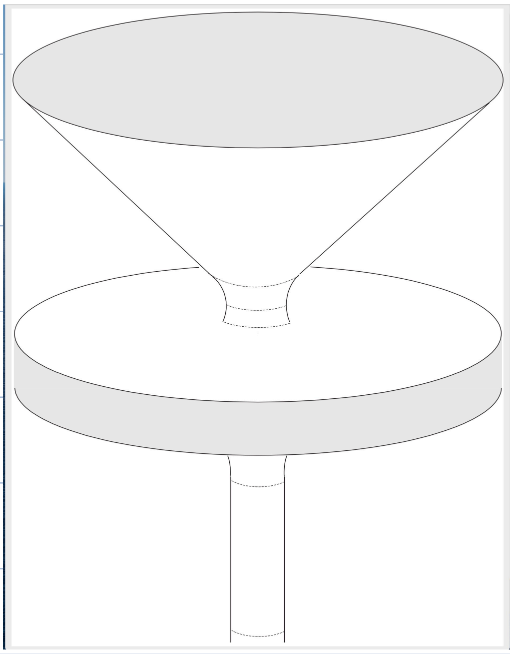


III Manifolds with finitely many (nice) ends.





$$\mathcal{R}^n = \mathbb{R}^n \times S^{N-n}$$
$$\mathcal{R}^m = \mathbb{R}^m \times S^{N-m}$$



$$M = M_1 \# M_2 \# \cdots \# M_k$$

(a_1) M is complete

(a_2) Each M_i is Harnack

$$p(t,x,y) \ ?$$

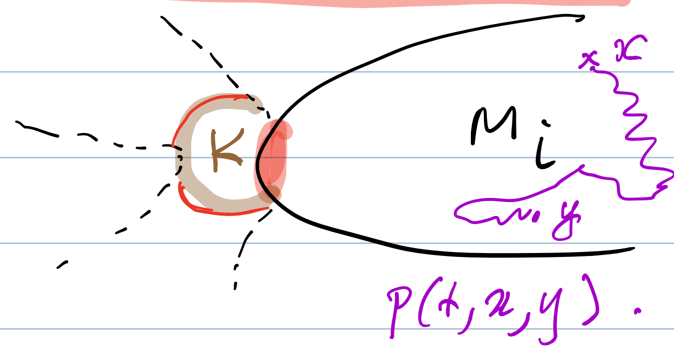
IV Key ideas (A) $M = M_1 \# M_2 \cdots \# M_k$

Estimate $p(t,x,y)$ using quantities that are determined by each end, separately.

which "quantities" ?

Most importantly: the Dirichlet heat kernel

$$p^{Dir}_{M_i}(t,x,y)$$

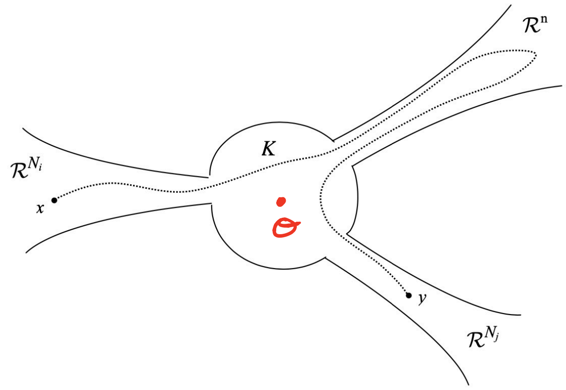


Hitting probability.

$$H(t,x) = \mathrm{Prob}_x\{\tau_K \leq t\}$$

Why can we estimate these quantities?

V Key ideas (B):



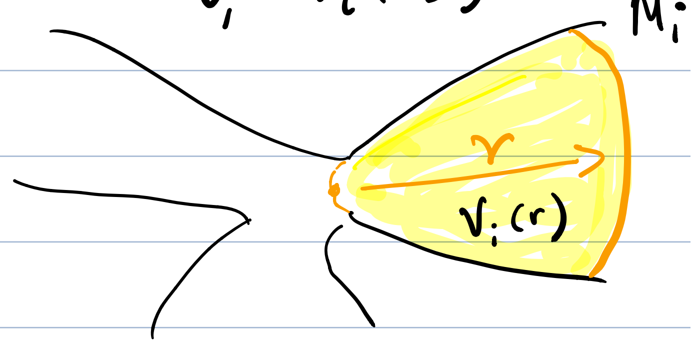
$p(t, o, o)$???

"Central heat kernel estimate"

This depends on the $\begin{cases} \text{parabolic / recurrent} \\ \text{non-parabolic / transience} \end{cases}$ nature of the ends.

Because of our main assumption (Harnack ends)

$$M_i \text{ is non-parabolic} \iff \int_1^{\infty} \frac{ds}{V_i(\sqrt{s})} < +\infty$$



VI Results:

Thm 1 (Grigor'yan & LSC)

All ends are non-parabolic

$$t \geq 1 : \quad p(t,o,o) \asymp \frac{1}{\min_{1\leq i \leq k}\{V_i(\sqrt{t})\}}$$

$\Rightarrow$ two-sided matching estimates for $p(t,x,y)$ for all t, x, y.

Thm 2 (Grigor'yan & LSC)

At least one non-parabolic end

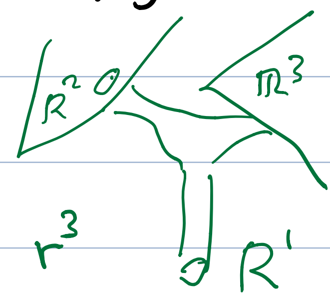
All parabolic ends are "RCA".

$$p(t,o,o) \simeq \frac{1}{\min_{1\le i\le k}\{\tilde{V}_i(\sqrt{t})\}}$$

$\tilde{V}_i(r) = V_i(r)\,\eta_i^2(r)$ where $\eta_i(r) = \max\left\{1, \int_1^{r^2} \frac{ds}{V_i(\sqrt{s})}\right\}$

$\Rightarrow$ two-sided matching estimates for $p(t,x,y)$ for all t, x, y.

Example: $\mathbb{R}^1 \# \mathbb{R}^2 \# \mathbb{R}^3$



$V_1(r) \simeq r$ $\quad V_2(r) \simeq r^2$ $\quad V_3(r) \simeq r^3$

$\eta_1(r) \simeq r$ $\quad \eta_2(r) \simeq \log r$ $\quad \eta_3(r) \simeq 1$

$\tilde{V}_1(r) \simeq r^3$ $\quad \tilde{V}_2(r) \simeq r^2(\log r)^2$ $\quad \tilde{V}_3(r) \simeq r^3$.

Thm 3 (Grigor'yan - Ishiwata - LSC)

All ends are parabolic and satisfy (RCA)

+ technical assumptions

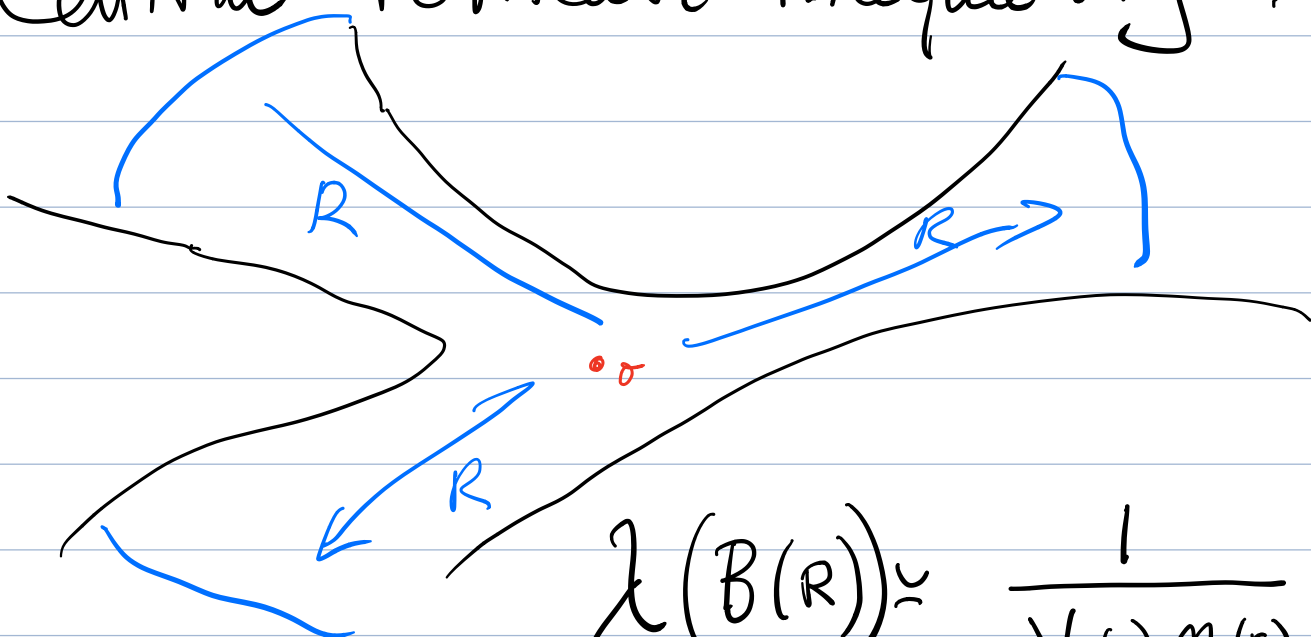
$$p(t,o,o) \simeq \frac{1}{\max_{1\le i\le k}\{V_i(\sqrt{t})\}}$$

$\Rightarrow$ two sided matching estimates for $p(t,x,y)$ for all $t>0$, $x,y\in M$.

Conjecture: $p(t,o,o) \simeq \dfrac{\min_{1\le i\le k} \eta_i^2(\sqrt{t})}{\min_{1\le i\le k}\{V_i(\sqrt{t})\,\eta_i^2(\sqrt{t})\}}$

VII

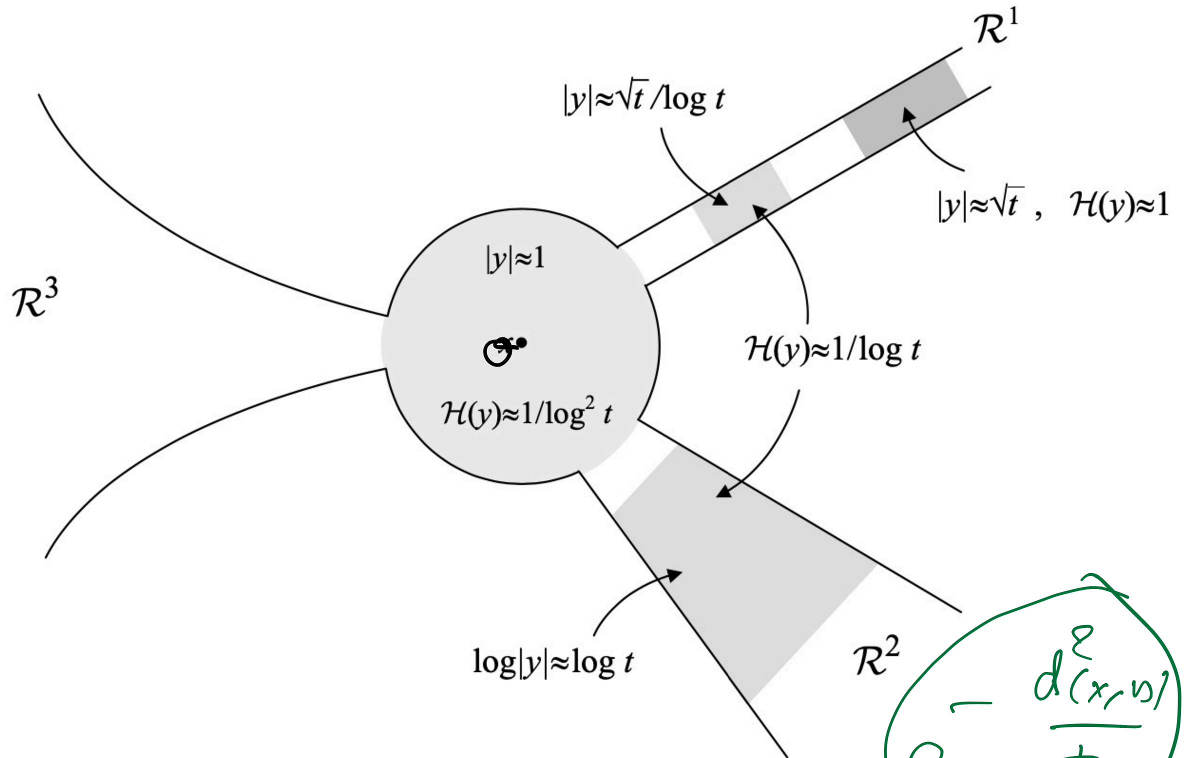
Central Poincaré inequality.



$$\lambda(B(R)) \simeq \frac{1}{V_*(r)\,\eta_*(r)}$$

where $*$ is the second largest end.

$\mathcal{R}^1 \# \mathcal{R}^2 \# \mathcal{R}^3$



$e^{-\frac{d^2(x,y)}{t}}$

$e^{-|x|^2/t}$

$e^{-|y|^2/t}$

$|x|$ = distance to the central part.

$$\mathcal{H}(y) = \frac{p(t,o,y)}{\sup_z p(t,o,z)}$$

Thank you!